Semi-Stochastic Coordinate Descent

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1. Problem

Many problems in data science (e.g. machine learning, optimization and statistics) can be cast as loss minimization problems of the form

$$\min_{x\in\mathbb{R}^d} f(x), \qquad \text{where} \qquad f(x)=\frac{1}{n}\sum_{i=1}^{n} f_i(x). \tag{P}$$

We assume that each individual function $f_i:\mathbb{R}^d\to\mathbb{R}$ is convex and has Lipschitz continuous partial gradients with constants $\{L_{ij}\}_j$. That is,

$$\|\nabla_j f_i(x)-\nabla_j f_i(y)\|\le L_{ij}\|x-y\|,\ \ \forall x,y\in\mathbb{R}^d.$$

Further we assume that $f:\mathbb{R}^d\to\mathbb{R}$ is μ-strongly convex:

$$f(y)\ge f(x)+\langle\nabla f(x),y-x\rangle+\frac{\mu}{2}\|y-x\|^2,\ \ \forall x,y\in\mathbb{R}^d.$$

2. Related Methods

Gradient Descent

Update:	$x_{k+1}=x_k-h\nabla f(x_k)$
# of iterations:	$\mathcal{O}(\kappa\log(1/\epsilon)$
Cost of 1 iteration:	$\mathcal{O}(n)$

Stochastic gradient descent (SGD)

Update:	1. Sample $i\in\{1,\dots,n\}:=[n]$
	2. $x_{k+1}=x_k-h_k\nabla f_i(x_k)$
# of iterations:	$\mathcal{O}(1/\epsilon)$
Cost of 1 iteration:	$\mathcal{O}(1)$

Coordinate descent (CD)

Update:	1. Sample $j\in\{1,\dots,d\}:=[d]$
	2. $x_{k+1}=x_k-h_j\nabla_j f(x_k)$
# of iterations:	$\mathcal{O}(\kappa\log(1/\epsilon))$
Cost of 1 iteration:	$\mathcal{O}(\omega)$ ω — degree of partial separability

Semi-stochastic gradient descent (S2GD) [3]

Update:	Outer loop:	Compute and store $\nabla f(x_k)$ $y_{k,0}=x_k$
	Inner loop:	1. Sample $i\in[n]$
		2. $y_{k,t+1}=y_{k,t}-h\left(\nabla f_i(y_{k,t})-\nabla f_i(x_k)+\nabla f(x_k)\right)$
		$x_{k+1}=y_{k,t_k}$
# of iterations:		$\lceil\log(1/\epsilon)\rceil$
Cost of 1 iteration:		$\mathcal{O}(n+\kappa)$

3. Goal

SGD type of methods are often seen as sampling *rows* of a data matrix. Conversely, CD methods usually sample *columns* of the data matrix.

The aim of this work is to develop a hybrid of S2GD and CD, which efficiently samples both rows and columns of data. The method computes a stochastic estimate of the partial gradient $\nabla_j f_i(x)$ with variance diminishing property and updates only one coordinate at each iteration.

4. The S2CD algorithm [1]

S2CD Algorithm — Semi-Stochastic Coordinate Descent

parameters: m (max # of stochastic steps per epoch); $h>0$ (stepsize parameter); $x_0\in\mathbb{R}^d$ (starting point); set $\beta=\sum_{t=1}^{m}(1-\mu h)^{m-t}$;
for $k=0,1,2,\dots$ **do**
  Compute and store $\nabla f(x_k)=\frac{1}{n}\sum_i\nabla f_i(x_k)$;
  Initialize the inner loop: $y_{k,0}\leftarrow x_k$;
  Let $t_k=t\in[m]$ with probability $(1-\mu h)^{m-t}/\beta$
  for $t=0$ to t_k-1 **do**
    Pick coordinate $j\in[d]$ with probability p_j
    Pick function index i from the set $\{i\ :\ L_{ij}>0\}$ with probability q_{ij}
    $y_{k,t+1}\leftarrow y_{k,t}-hp_j^{-1}(\nabla_j f(x_k)+\frac{1}{nq_{ij}}\left(\nabla_j f_i(y_{k,t})-\nabla_j f_i(x_k)\right))e_j$;
  end for
  Reset the starting point: $x_{k+1}\leftarrow y_{k,t_k}$;
end for

The selection probability $\{p_j\}$ and $\{q_{ij}\}$ in S2CD are defined by:

$$p_j:=\frac{\sum_{i=1}^{n}\omega_i L_{ij}}{\hat{L}}, \qquad q_{ij}:=\frac{\omega_i L_{ij}}{\sum_{i=1}^{n}\omega_i L_{ij}},$$

where

$$\omega_i:=|\{j:L_{ij}\neq 0\}|, \qquad \hat{L}:=\frac{1}{n}\sum_{j=1}^{d}\sum_{i=1}^{n}\omega_i L_{ij}.$$

Special cases:

- When $n=1$: S2CD reduces to a stochastic CD algorithm with importance sampling for the selection of $j\in[d]$.
- It is possible to extend S2CD to the case when coordinates are replaced by (nonoverlapping) blocks of coordinates. In such a setting, when all the variables form a single block, S2CD reduces to S2GD but with importance sampling for the selection of $i\in[n]$, as in [5].

Theorem 1

If $0<h<1/(2\hat{L})$ and m is sufficiently large so that

$$c:=\frac{(1-\mu h)^m}{(1-(1-\mu h)^m)(1-2\hat{L}h)}+\frac{2\hat{L}h}{1-2\hat{L}h}<1,$$

then for all $k\ge 0$ we have:

$$\mathbf{E}[f(x_k)-f(x_*)]\le c^k\,\mathbf{E}[f(x_k)-f(x_*)].$$

Corollary: *Let $\hat{\kappa}:=\hat{L}/\mu$. If we run the algorithm S2CD with stepsize h and m set as*

$$h=\frac{1}{(4e+2)\hat{L}}, \qquad m\ge(4e+2)\log(2e+2)\,\hat{\kappa},$$

then for all $k\ge\lceil\log(1/\epsilon)\rceil$,

$$\mathbf{E}[f(x_k)-f(x_*)]\le\epsilon(f(x_0)-f(x_*)).$$

5. Complexity & comparison

Definition
We let $\mathcal{C}_{grad}$ be the average cost of evaluating the stochastic gradient ∇f_i and $\mathcal{C}_{pd}$ be the average cost of evaluating the stochastic partial derivative $\nabla_j f_i$.

S2CD complexity
The total work of S2CD can be written as

$$\mathcal{O}\left(\left(n\mathcal{C}_{grad}+\hat{\kappa}\mathcal{C}_{pd}\right)\log(1/\epsilon)\right).$$

The complexity results of methods such as S2GD/SVRG [3, 2, 5] and SAG/SAGA [4, 6] — in a similar but not identical setup to ours (some of these papers assume f_i to be L_i-smooth) — can be written in a similar form:

$$\mathcal{O}\left(\left(n\mathcal{C}_{grad}+\kappa\mathcal{C}_{grad}\right)\log(1/\epsilon)\right),$$

where $\kappa=L/\mu$ and either $L=L_{max}$ ([4, 2, 3, 6]), or $L=L_{avg}:=\frac{1}{n}\sum_{i,j}L_{ij}$ ([5]).

The difference between our result and existing results is in the term $\hat{\kappa}\mathcal{C}_{pd}$ – previous results have $\kappa\mathcal{C}_{grad}$ in that place. This difference constitutes a trade-off: while $\hat{\kappa}\ge\kappa$, we clearly have $\mathcal{C}_{pd}\le\mathcal{C}_{grad}$. The comparison of the quantities $\kappa\mathcal{C}_{grad}$ and $\hat{\kappa}\mathcal{C}_{pd}$ is not straightforward and is problem dependent.

Conclusion
S2CD can be both better or worse than S2GD/SVRG/SAG/SAGA, depending on whether the increase of the condition number from κ to $\hat{\kappa}$ can or can not be compensated by the decrease of the derivative evaluation from $\mathcal{C}_{grad}$ to $\mathcal{C}_{pd}$.

6. References

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